ON THE HYDRODYNAMIC PRESSURE ON A DAM CAUSED BY ITS APERIODIC OR IMPULSIVE VIBRATIONS AND VERTICAL VIBRATIONS OF THE EARTH SURFACE

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This paper is concerned with the problem of the distribution along a dam of hydrodynamic pressure caused by aperiodic or impulsive vibrations of the dam, and vertical vibrations of the ground below the liquid. The results show that the vertical vibrations of the earth surface have a significant influence upon the loading of the dam during a strong as well as during a destructive earthquake. Formulas for the distribution of the dynamic fluid pressure along the dam are derived.

The problem of the dynamic fluid pressure on a dam, caused by its periodic vibrations, for instance $V = V_0 \cos \omega t$, was studied in [1-4], where V_0 was the velocity amplitude of the vibrating dam. The problem of surface waves on a fluid which appear due to a periodic surface, or internal, pressure system was studied in [5-7].

1. We shall analyze the problem of the dynamic pressure of the fluid on the dam, caused by vibrations of the earth surface with a velocity $V(t)$, which lies in the plane x, y and is inclined at an angle ϑ to the horizon.

Assume that in rectangular coordinates x, y, z the dam and the earth surface are located at $x = U_1(t)$ and $y = U_2(t) - h$, respectively. The part of the space that is bounded by $x \geqslant U_1(t)$, $U_2(t) - h \leqslant y \leqslant U_2(t)$, and $-\infty \leqslant z \leqslant \infty$ is filled with fluid.

Let us assume that the surface of the liquid is initially at rest. When the velocity potential of the fluid is denoted by $\phi(x, y, t)$ the initial and boundary conditions of the problem become

$$\frac{\partial\varphi(x,\,0,\,0)}{\partial t}=0, \qquad \frac{\partial\varphi(0,\,y,\,0)}{\partial x}=V_1(0), \qquad \frac{\partial\varphi(x,\,-h,\,0)}{\partial y}=V_2(0) \quad (1.1)$$

$$\frac{\partial\varphi}{\partial x}=V_1(t) \quad \text{at } x=U_1(t)=\int_0^t V_1(\tau)\,d\tau \qquad (V_1(t)=V(t)\cos\vartheta) \quad (1.2)$$

$$\frac{\partial\varphi}{\partial y}=V_2(t) \quad \text{at } y=U_2(t)-h \quad (1.3)$$

$$\frac{\partial^2\varphi}{\partial t^2}+g\frac{\partial\varphi}{\partial y}=0 \quad \text{at } y=U_2(t)=\int_0^t V_2(\tau)\,d\tau \qquad (V_2(t)=V(t)\sin\vartheta) \quad (1.4)$$

The fluid velocity potential, which has to satisfy the Laplace equation $\Delta\phi=0$, can be written as follows:

$$\varphi(x,\,y,\,t)=\iint_0^\infty [B(\omega,\,k)\cosh k(Y+h)+D(\omega,\,k)\sinh kY]\cos kX\cos\omega t\,d\omega\,dk+$$
$$+\iint_0^\infty A(\omega,\alpha)\sin\alpha Y e^{-\alpha X}\cos\omega t\,d\omega\,d\alpha \qquad \begin{pmatrix} X=x-U_1(t)\\ Y=y-U_2(t)\end{pmatrix} \quad (1.5)$$

Here $A(\omega,\,a)$, $B(\omega,\,k)$, and $D(\omega,\,k)$ are arbitrary functions, and the function $\phi(x,\,y,\,t)$ is determined so that

$$\varphi=\varphi(x,\,y,\,t) \quad \text{at } x\geqslant U_1(t)\ (-h\leqslant Y\leqslant 0), \qquad \varphi=0 \quad \text{at } x<U_1(t)$$

The boundary condition (1.2) will be satisfied if we choose

$$-\iint_0^\infty \alpha A(\omega,\,\alpha)\sin\alpha Y\cos\omega t\,d\omega\,d\alpha=V_1(t) \quad (1.6)$$

Introduce a new variable $\zeta=Y$ and rewrite (1.6) in the form

$$-\iint_0^\infty \alpha A(\omega,\,\alpha)\sin\alpha\zeta\cos\omega t\,d\omega\,d\alpha=V_1(t)f(\zeta) \quad (1.7)$$

where

$$f(\zeta)=1 \quad \text{at } -h\leqslant\zeta\leqslant 0, \qquad f(\zeta)=0 \quad \text{at } -h>\zeta>0$$

Using a Fourier expansion we obtain from the integral equation (1.7)

$$A(\omega,\,\alpha)=\frac{4(1-\cos\alpha h)}{\pi^2\alpha^2}G_1(\omega) \qquad \left(G_1(\omega)=\int_0^\infty V_1(\tau)\cos\omega\tau\,d\tau\right) \quad (1.8)$$

Because of the initial condition (1.3) we have

$$\int_0^\infty\left[\int_0^\infty D(\omega,k)\,k\cosh\,kh\cos kX dk+\int_0^\infty A(\omega,\alpha)\cos\alpha h e^{-\alpha X}\alpha d\alpha\right]\cos\omega t\,d\omega=V_2(t)$$
$$(1.9)$$

From this

$$D(\omega,\ k) = -\frac{2}{\pi}\int_0^\infty \frac{A(\omega,\ \alpha)\,\alpha^2\cos\alpha h}{k(\alpha^2+k^2)\cosh kh}\,d\alpha + \frac{2}{\pi}\,\frac{G_2(\omega)\,\delta(k)}{k\cosh kh} \tag{1.10}$$

Here

$$\delta(k) = \frac{2}{\pi}\lim_{l\to\infty}\frac{\sin kl}{k} \quad \left(\text{Dirac delta function}\right), \qquad G_2(\omega) = \int_0^\infty V_2(\tau)\cos\omega\tau\,d\tau$$

From boundary condition (1.4) we determine the unknown function

$$B(\omega,\ k) = -\frac{kgD(\omega,\ k)}{kg\sinh kh - \omega^2\cosh kh} - \frac{2g}{\pi}\int_0^\infty \frac{A(\omega,\ \alpha)\,\alpha^2 d\alpha}{(\alpha^2+k^2)\,(kg\sinh kh - \omega^2\cosh kh)} \tag{1.11}$$

After substitution of Expressions (1.8), (1.10), and (1.11) into (1.5), we obtain the sought-for velocity potential $\phi(x,\ y,\ t)$.

1) In order to find the dynamic fluid pressure on the dam caused by its aperiodic vibrations we shall assume that

$$V(t) = V_0 e^{-\lambda t} \qquad (\lambda = \xi + i\eta) \tag{1.12}$$

where V_0, ξ and η are real constants. Here the velocities of the vibrating dam $V_1(t)$ and the earth surface below the fluid $V_2(t)$ will be respectively

$$V_1(t) = V_1 e^{-\lambda t}, \qquad V_2(t) = V_2 e^{-\lambda t} \qquad (V_1 = V_0\cos\vartheta,\ V_2 = V_0\sin\vartheta) \tag{1.13}$$

$$U(t) = \int_0^t V(\tau)\,d\tau = U_0(1 - e^{-\lambda t}) \qquad \left(U_0 = \frac{V_0}{\lambda}\right)$$

From this, the displacement of the dam $V_1(t)$ and the earth surface $V_2(t)$ will be

$$U_1(t) = U_1(1 - e^{-\lambda t}), \quad U_2(t) = U_2(1 - e^{-\lambda t}) \quad (U_1 = U_0\cos\vartheta,\ U_2 = U_0\sin\vartheta) \tag{1.14}$$

We find from (1.12), taking into account (1.8) and (1.10), that

$$G_1(\omega) = \frac{\lambda V_1}{\lambda^2+\omega^2}, \qquad G_2(\omega) = \frac{\lambda V_2}{\lambda^2+\omega^2} \tag{1.15}$$

Let us introduce the following notation:

$$\Psi(\alpha, k, \omega) = (1 - \cos\alpha h) G_1(\omega) / (\alpha^2 + k^2)(kg \sinh kh - \omega^2 \cosh kh)$$

$$S_1 = \iiint_0^\infty \frac{\cos\alpha h}{\cosh kh} \Psi(\alpha, k, \omega) \cosh k(Y+h) \omega \sin\omega t \, d\alpha \, dk \, d\omega$$

$$S_2 = \iiint_0^\infty \Psi(\alpha, k, \omega) \cosh k(Y+h) \omega \sin\omega t \, d\alpha \, dk \, d\omega$$

$$S_3 = \iiint_0^\infty \frac{\cos\alpha h}{\cosh kh} \Psi(\alpha, k, \omega) k \sinh k(Y+h) \cos\omega t \, d\alpha \, dk \, d\omega \qquad (1.16)$$

$$S_4 = \iiint_0^\infty \Psi(\alpha, k, \omega) k \sinh k(Y+h) \cos\omega t \, d\alpha \, dk \, d\omega$$

When we differentiate $\phi(x, y, t)$ with respect to t for $t > 0$, $x = U_1(t)$, we have

$$\frac{\partial\varphi}{\partial t} = \frac{8}{\pi^3} \iiint_0^\infty \frac{(1-\cos\alpha h)\cos\alpha h}{k(\alpha^2+k^2)\cosh kh} G_1(\omega) [\omega \sin\omega t \sinh kY +$$

$$+ V_2 e^{-\lambda t} k \cosh kY \cos\omega t] \, d\alpha \, dk \, d\omega - \frac{8g}{\pi^3} [S_1 - S_2 + V_2 e^{-\lambda t}(S_3 - S_4)] +$$

$$+ U_2 g (e^{-\lambda t} - 1) - \lambda V_2 Y e^{-\lambda t} - V_2^2 e^{-2\lambda t} -$$

$$- \frac{4}{\pi^2} \iint_0^\infty \frac{(1-\cos\alpha h)}{\alpha^2} G_1(\omega) [\omega \sin\omega t \sin\alpha Y +$$

$$+ \alpha (V_2 \cos\alpha Y - V_1 \sin\alpha Y) e^{-\lambda t} \cos\omega t] \, d\alpha \, d\omega \quad (1.17)$$

First let us study S_1. Integration with respect to ω yields

$$T = \lambda V_1 \int_0^\infty \frac{\omega \sin\omega t \, d\omega}{(\omega^2+\lambda^2)(kg \sinh kh - \omega^2 \cosh kh)} = \frac{\pi\lambda}{2} V_1 \frac{e^{-\lambda t} - \cos\sqrt{kg \tanh kh}\, t}{kg \sinh kh + \lambda^2 \cosh kh} \quad (1.18)$$

Introduce the notation $M_1 = \sqrt{[kg \tanh(kh)]}$. The substitution of (1.18) into the expression for S_1 in (1.16) yields

$$S_1 = \frac{\pi\lambda}{2} V_1 (R_1 e^{-\lambda t} - R_2) \qquad (1.19)$$

where

$$R_1 = \int\int_0^\infty \frac{(1-\cos\alpha h)\cos\alpha h \cosh k(Y+h)\, d\alpha\, dk}{(\alpha^2+k^2)(kg\sinh kh+\lambda^2\cosh kh)\cosh kh}$$

$$R_2 = \int\int_0^\infty \frac{(1-\cos\alpha h)\cos\alpha h \cosh k(Y+h)\cos M_1 t}{(\alpha^2+k^2)(kg\sinh kh+\lambda^2\cosh kh)\cosh kh}\, d\alpha\, dk$$

Since the integrals (1.19) converge uniformly with respect to a and k, we may change the order of integration, i.e. we shall integrate at first with respect to k. Rewrite R_1 in the following form:

$$R_1 = \int_0^\infty (1-\cos\alpha h)\cos\alpha h\, d\alpha R_1^* \qquad (1.20)$$

$$\left(R_1^* = \int_0^\infty \frac{\cosh k(Y+h)\, dk}{(\alpha^2+k^2)(kg\sinh kh+\lambda^2\cosh kh)\cosh kh}\right)$$

R_1^* will be calculated by the use of the theory of residues. In the complex plane we have two roots $\pm i a$ and an infinite number of roots $\pm\, i\, m\pi/2h$, where $m = 1, 3, 5, \ldots,$ for the equations $a^2 + k^2 = 0$ and $\cosh(kh) = 0$, respectively. In order to find the roots of the transcendental equation $kg \sinh(kh) + \lambda^2 \cosh(kh) = 0$ we shall introduce a new variable $kh = \gamma$ and transform this equation to

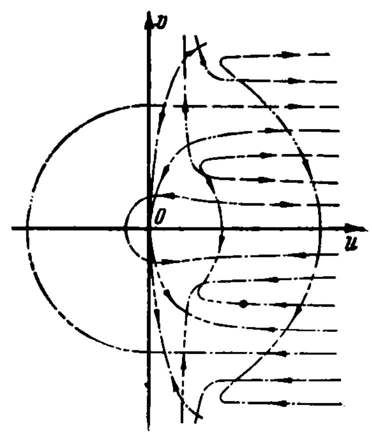

Fig. 1.

$$\gamma\, \mathrm{th}\, \gamma = \mu \qquad (\mu = -\lambda^2 h/g = (\eta^2 - \xi^2 - i2\eta\xi)\, h/g) \qquad (1.21)$$

Using the conformal transformation

$$w = f(z') = z' \tanh z' \qquad (w = u + iv,\ z' = x' + iy') \qquad (1.22)$$

we have

$$u = \frac{x' \sinh 2x' - y' \sin 2y'}{\cos 2y' + \cosh 2x'}, \qquad v = \frac{y' \sinh 2x' + x' \sin 2y'}{\cos 2y' + \cosh 2x'} \qquad (1.23)$$

From Formula (1.20) and Figs. 1 and 2, it can be seen that the mapping (1.22) transforms the parallel lines $\pm\, n\pi/4$, where $n = 1, \ldots, 8$, in the z'-plane into curves in the w-plane. With the aid of these figures, with a known μ, we shall find the roots of Equation (1.21) in the z'-plane, which corresponds to point μ in the w-plane. Besides, we obtain by means of successive approximations from Formulas (1.21) and (1.23) the unknown γ with the necessary degree of accuracy. We see from Figs. 1 and

2 and Formula (1.23) that Equation (1.21) has the following roots: when μ is a complex constant there are several complex roots (two, four, etc.); for $\mu < 0$ there is an infinite number of imaginary roots; for $\mu > 0$ there are two real roots and an infinite number of imaginary roots; when μ is an imaginary constant there are several complex roots. The equality $\xi = \eta$ corresponds to the last case.

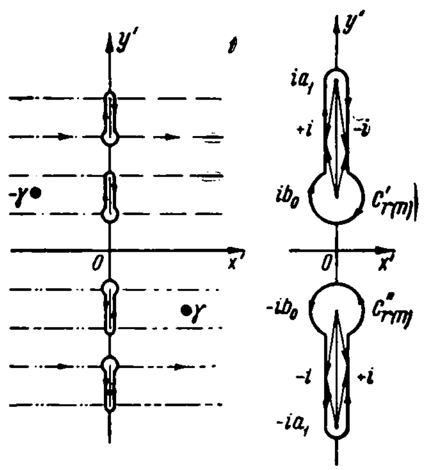

Fig. 2.

As an example, let us assume that $\xi < \eta$, and that point μ is located as shown in Fig. 1.

After a number of integrations we find

$$R_1 = -i\frac{\pi^2 h}{2}\frac{(1-\cosh\gamma)\quad[\gamma(Y+h)/h]}{g[\gamma^2-(1+\mu)\mu]\cosh^2\gamma}e^{-\gamma}+\frac{\pi}{2}\int_0^\infty\frac{(1-\cos\alpha h)\cos\alpha(Y+h)\,d\alpha}{\alpha(\lambda^2\cos\alpha h-\alpha g\sin\alpha h)}+$$

$$+\pi\int_0^\infty\sum_{m=1,3}^\infty\frac{\cos C(Y+h)\cos\alpha h}{gC(C^2-\alpha^2)h}(1-\cos\alpha h)\,d\alpha \qquad \begin{pmatrix}\gamma = p - iq\\ C = m\pi/2h\end{pmatrix} \qquad (1.24)$$

Here p and q are real constants.

Returning to the computation of R_2, we analyze the integral

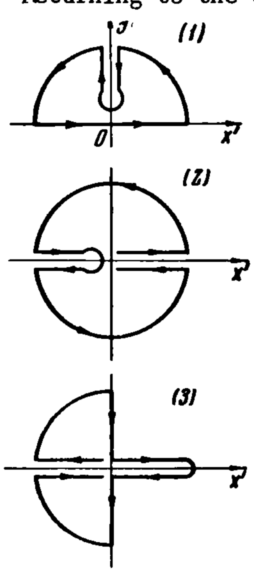

Fig. 3.

$$R_2^* = \int_0^\infty\frac{\cosh k(Y+h)\cos M_1 t\,dk}{(\alpha^2+k^2)(kg\sinh kh+\lambda^2\cosh kh)\cosh kh} \qquad (1.25)$$

For the evaluation of this integral, we choose three auxiliary functions

$$F_1(z') = \frac{\cosh z'(Y+h)\exp[iM_1(z')t]}{(\alpha^2+z'^2)(z'g\sinh z'h+\lambda^2\cosh z'h)\cosh z'h}$$

$$F_2(z') = \frac{\cos z'(Y+h)\exp[-M_2(z')t]}{(\alpha^2-z'^2)(\lambda^2\cos z'h-z'g\sin z\,h)\cos z'h} \qquad (1.26)$$

$$F_3(z') = \frac{\cos z'(Y+h)\exp[M_2(z')t]}{(\alpha^2-z'^2)(\lambda^2\cos z'h-z'g\sin z'h)\cos z'h}$$

and the corresponding contours, shown in Fig. 3, where contour (1) lies in plane 1 (Fig. 2) and contours (2) and (3) lie in plane 1′, which can be obtained from plane 1 by a mapping of (x', y') onto $(-y', x')$ and a 90° clockwise rotation.

Here we have two roots $\pm iy$ in the transformed plane corresponding to the equation $\lambda^2 \cos z'h - z'g \sin z'h = 0$.

Before we go into the contour integration we shall study the multi-valued functions

$$M_1(z') = \sqrt{z'g \tanh z'h}, \qquad M_2(z') = \sqrt{z'g \tan z'h} \tag{1.27}$$

By expanding tanh $(z'h)$ and tan $(z'h)$ in terms of infinite products

$$\begin{aligned} \tanh\chi &= \chi \prod_{n=1}^{\infty}\left(1+\frac{\chi^2}{n^2\pi^2}\right) \Big/ \prod_{n=0}^{\infty}\left(1+\frac{4\chi^2}{(2n+1)^2\pi^2}\right) \\ \tan\chi &= \chi \prod_{n=1}^{\infty}\left(1-\frac{\chi^2}{n^2\pi^2}\right) \Big/ \prod_{n=0}^{\infty}\left(1-\frac{4\chi^2}{(2n+1)^2\pi^2}\right) \end{aligned} \tag{1.28}$$

where $\chi = z'h$, we obtain

$$\begin{aligned} M_1(z') &= z'C_0\sqrt{\frac{z'-ia_1}{z'-ib_0}}\sqrt{\frac{z'+ia_1}{z'+ib_0}}\sqrt{\frac{z'-ia_2}{z'-ib_1}}\cdots \\ M_2(z') &= z'C_0\sqrt{\frac{z'-a_1}{z'-b_0}}\sqrt{\frac{z'+a_1}{z'+b_0}}\sqrt{\frac{z'-a_2}{z'-b_1}}\cdots \end{aligned} \tag{1.29}$$

where C_0, b_0, a_1, b_1, a_2 are real constants, and $a_{n+1} = (n+1)\pi/h$, $b_n = (2n+1)\pi/2h$. We see from this that the function $M_1(z')$ has an infinite number of branch points on the imaginary axis, and $M_2(z')$ has similar points, except that they appear on the real axis.

Plane z' will be cut as shown in Fig. 2. After that, the functions $M_1(z')$ and $M_2(z')$ will be single-valued inside the corresponding contour in the multiply-connected region. Let

$$M_1^*(z') = M_1(z')/z'$$

The values of the arguments on the left and the right edge of the cuts along the positive imaginary axis will be respectively

$$\arg M_1^*(z') = \tfrac{1}{2}\left[-\tfrac{1}{2}\pi - \left(\tfrac{1}{2}\pi - 2\pi\right) + 0 + 0 + \cdots\right] = \tfrac{1}{2}\pi$$

$$\arg M_1^*(z') = \tfrac{1}{2}\left(-\tfrac{1}{2}\pi - \tfrac{1}{2}\pi + 0 + 0 + \cdots\right) = -\tfrac{1}{2}\pi$$

Therefore the function $M_1^*(z')$ along the left and the right edges has the multipliers $+i$ and $-i$, respectively, in front of the root. In an analogous manner we determine arg $M_1^*(z')$ along the corresponding edges of the segments located along the negative imaginary axis (Fig. 2).

Note that during the integrations along the contours (1) and (3) the paths followed the directions shown by the arrows in Fig. 2, and along contour (2) the integration proceeded clockwise. By applying the residue theorem we obtain

$$\oint_{(1)I} F_1(z')\,dz' = 2R_2^* + iN_1 + iN_2 + \sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C'_{r(n)}} F_1(z')\,dz' = -iN^* e^{-\lambda t} + f_1^*(\alpha,\ Y,\ t) \tag{1.30}$$

$$\oint_{(2)I'} F_2(z')\,dz' = 2N_1 + 2N_2 + H_1 + \sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C'_{r(n)}} F_2(z')\,dz' = N^*(e^{\lambda t} - e^{-\lambda t}) + f_2^*(\alpha,\ Y,\ t)$$

$$\oint_{(3)I'} F_3(z')\,dz' = -i2R_2^* + H_2 + \sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C''_{r(n)}} F_3(z')\,dz' + f_3^*(\alpha,\ Y,\ t) = -N^* e^{-\lambda t}$$

Here

$$N_1 = \sum_{n=0}^{\infty} \int_{a_{n+1}}^{b_n} \frac{\cos k(Y+h)\exp(iM_2 t)\,dk}{(\alpha^2 - k^2)(\lambda^2 \cos kh - kg \sin kh)\cos kh}$$

$$N_2 = \sum_{n=0}^{\infty} \int_{b_n}^{a_{n+1}} \frac{\cos k(Y+h)\exp(-i M_2 t)\,dk}{(\alpha^2 - k^2)(\lambda^2 \cos kh - kg \sin kh)\cos kh}$$

$$N^* = \frac{2\pi\gamma h^2 \cosh[\gamma(Y+h)/h]}{g(\alpha^2 h^2 + \gamma^2)\,\Gamma(\gamma,\ \mu)\cosh^2\gamma}, \quad M_2 = \sqrt{kg \tan kh}, \quad \Gamma(\gamma,\ \mu) = \gamma^2 - (1+\mu)\mu$$

where $f_\nu^*(a,\ Y,\ t)$ are some functions of a, Y and t, where $\nu = 1,\ 2,\ 3$.

Of course, all integrals along the large circles are equal to zero when their radii tend to ∞. Let us study the integrals under the summation signs in Formulas (1.30) and integrate them along the small circles $C'_{r(n)}$ and $C''_{r(n)}$.

As the radii of the circles tend to zero we have

$$\sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C'_{r(n)}} F_1(z')\,dz' \tag{1.31}$$

$$= -i \sum_{n=0}^{\infty} C_n \lim_{r\to 0} \int_{\pi/2}^{-3\pi/2} \exp\left[-\frac{C_n^{(1)}}{\sqrt{r}}(1-i)\exp\left(\frac{-i\beta}{2}\right)\right] d\beta$$

$$= -i2 \sum_{n=0}^{\infty} C_n \lim_{r\to 0} \int_{\pi/2}^{-\pi/2} \exp\left[-\frac{C_n^{(2)}}{\sqrt{r}}(\cos\varkappa - i\sin\varkappa)\right] d\varkappa \to 0$$

Here C_n, $C_n^{(1)}$, and $C_n^{(2)}$ are some positive constants, β and κ are arguments. In the same manner one can prove that

$$\sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C'_{r(n)}} F_2(z')\,dz' \to 0, \qquad \sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C''_{r(n)}} F_3(z')\,dz' \to 0 \tag{1.32}$$

Note that

$$H_1 = \sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C''_{r(n)}} F_2(z')\,dz' = i2 \sum_{n=0}^{\infty} C_n \lim_{r\to 0} \int_{\pi/2}^{-\pi/2} \exp\left[\frac{C_n^{(1)}}{\sqrt{r}}(\cos\varkappa - i\sin\varkappa)\right] d\varkappa \tag{1.33}$$

$$H_2 = \sum_{n=0}^{\infty} \lim_{r\to 0} \int_{C'_{r(n)}} F_3(z')\,dz' =$$

$$= -i\sum_{n=0}^{\infty} C_n \lim_{r\to 0} \int_{2\pi}^{0} \exp\left[i\frac{C_n^{(1)}}{\sqrt{r}}\exp\left(\frac{-i\beta}{2}\right)\right] d\beta = -H_1$$

From the system of equations (1.30) and the relations (1.31), (1.32), and (1.33) we find the required integral

$$R_2^* = -iN^* e^{\lambda t}/2 + f^*(\alpha,\ Y,\ t) \tag{1.34}$$

After substituting (1.34) and (1.24) into Formula (1.19) and carrying out the R_2-integration with respect to a, we obtain

$$S_1 = i\frac{\pi^3}{2}\lambda h e^{-\gamma} V_1 \frac{(1-\cosh\gamma)\sinh\lambda t}{g\Gamma(\gamma,\ \mu)\cosh^2\gamma}\cosh\left[\frac{\gamma}{h}(Y+h)\right] + e^{-\lambda t}\int_0^{\infty} f_1(\alpha,\ Y)\,d\alpha +$$

$$+\int_0^{\infty} f_2(\alpha,\ Y,\ t)\,d\alpha + \frac{\pi^2}{2}\lambda e^{-\lambda t} V_1 \int_0^{\infty} \sum_{m=1,\,3}^{\infty} \frac{\cos C(Y+h)\cos\alpha h}{gC(C^2-\alpha^2)h}(1-\cos\alpha h)\,d\alpha \tag{1.35}$$

In the same manner we find (1.36)

$$S_2 = i\frac{\pi^3}{2}\lambda h V_1 \frac{\cosh[\gamma(Y+h)/h]}{g\Gamma(\gamma,\ \mu)\cosh\gamma}(1-e^{-\gamma})\sinh\lambda t + e^{-\lambda t}\int_0^\infty f_1(\alpha,\ Y)d\alpha + \int_0^\infty f_2(\alpha, Y, t)d\alpha$$

$$S_3 = -i\frac{\pi^3}{2}\gamma e^{-\gamma}V_1\frac{(1-\cosh\gamma)\cosh\lambda t}{g\Gamma(\gamma,\ \mu)\cosh^2\gamma}\sinh\left[\frac{\gamma}{h}(Y+h)\right] + e^{-\lambda t}\int_0^\infty f_3(\alpha,\ Y)\,d\alpha +$$

$$+\int_0^\infty f_4(\alpha,\ Y,\ t)\,d\alpha + \frac{\pi^2}{2}e^{-\lambda t}V_1\int_0^\infty \sum_{m=1,3}^{\infty}\frac{\sin C(Y+h)\cos\alpha h}{gh(\alpha^2-C^2)}(1-\cos\alpha h)\,d\alpha$$

$$S_4 = -i\frac{\pi^3}{2}\gamma V_1\frac{\sinh[\gamma(Y+h)/h]}{g\Gamma(\gamma,\ \mu)\cosh\gamma}(1-e^{-\gamma})\cosh\lambda t + e^{-\lambda t}\int_0^\infty f_3(\alpha,\ Y)d\alpha + \int_0^\infty f_4(\alpha, Y, t)d\alpha$$

After substituting Formulas (1.35) and (1.36) into (1.17) and evaluating the remaining integrals, we obtain finally

$$\frac{\partial\varphi}{\partial t} = i4\mu\lambda h V_1\frac{\cosh[\gamma(Y+h)/h]}{\gamma\Gamma(\gamma,\ \mu)\cosh\gamma}\sinh\lambda t - i4\mu e^{-\lambda t}V_1V_2\frac{\sinh[\gamma(Y+h)/h]}{\Gamma(\gamma,\ \mu)\cosh\gamma}\cosh\lambda t -$$

$$-\lambda V_2 Y e^{-\lambda t} - (V_1^2+V_2^2)e^{-2\lambda t} + U_2 g(e^{-\lambda t}-1) \qquad (1.37)$$

for $t > 0$, $Y < 0$, and $x = U_1(t)$. In the same manner we find for $t > 0$, $Y < 0$, and $x = U_1(t)$

$$\frac{\partial\varphi}{\partial y} = i4\mu V_1\frac{\sinh[\gamma(Y+h)/h]}{\Gamma(\gamma,\ \mu)\cosh\gamma}\cosh\lambda t + V_2 e^{-\lambda t} \qquad (1.38)$$

It is easily shown that

$$\frac{\partial\varphi}{\partial x} = V_1 e^{-\lambda t} \qquad (1.39)$$

In order to determine the dynamic fluid pressure at $t = 0$, we turn to Formula (1.17). After integration with respect to ω and setting $t = 0$, we obtain

$$\frac{\partial\varphi}{\partial t} = \frac{4}{\pi^2}\lambda V_1\iint_0^\infty\frac{(1-\cos\alpha h)\cos\alpha h}{k(\alpha^2+k^2)\cosh kh}\sinh kY\,d\alpha\,dk +$$

$$+\frac{4}{\pi^2}V_1V_2\iint_0^\infty\frac{(1-\cos\alpha h)\cos\alpha h}{(\alpha^2+k^2)\cosh kh}\cosh kY\,d\alpha\,dk -$$

$$-\frac{4g}{\pi^2}V_1V_2\iint_0^\infty\frac{(1-\cos\alpha h)\,k\sinh k(Y+h)}{(\alpha^2+k^2)(kg\sinh kh+\lambda^2\cosh kh)}\left(\frac{\cos\alpha h}{\cosh kh}-1\right)d\alpha\,dk - \lambda V_2 Y - V_2^2 - V_1^2 -$$

$$-\frac{2}{\pi}V_1\int_0^{\infty}\frac{(1-\cos\alpha h)}{\alpha^2}(\lambda\sin\alpha Y+V_2\alpha\cos\alpha Y)\,d\alpha \qquad (1.40)$$

At first we carry out the double integration in (1.40) with respect to a in the first integral, and in the second and third double integrals we integrate first with respect to k. We find at $t = 0$, $y = 0$, and $x = 0$ that

$$\frac{\partial\varphi}{\partial t}=-\frac{2\lambda}{h}V_1\sum_{m=1,3}^{\infty}\frac{\sin CY}{C^2}-i2\mu V_1V_2\frac{\sinh[\gamma(Y+h)/h]}{\Gamma(\gamma,\mu)\cosh\gamma}-\lambda V_2Y-V_1^2-V_2^2 \qquad (1.41)$$

Note that in the calculation of $\partial\phi/\partial y$ we integrate all double integrals with respect to k first. Integration yields

$$\frac{\partial\varphi}{\partial y}=i2\mu V_1\frac{\sinh[\gamma(Y+h)/h]}{\Gamma(\gamma,\mu)\cosh\gamma}+V_2 \qquad \left(\frac{\partial\varphi}{\partial x}=V_1\right) \qquad (1.42)$$

2) Now let us study the dynamic pressure on the dam caused by its impulsive action on the liquid. Let us assume that

$$V(t)=V_0e^{-\xi t} \qquad (1.43)$$

Thus, actually, after the change of λ to ξ Formulas (1.12) to (1.19) hold also for this case. Note that in the evaluation of the integrals

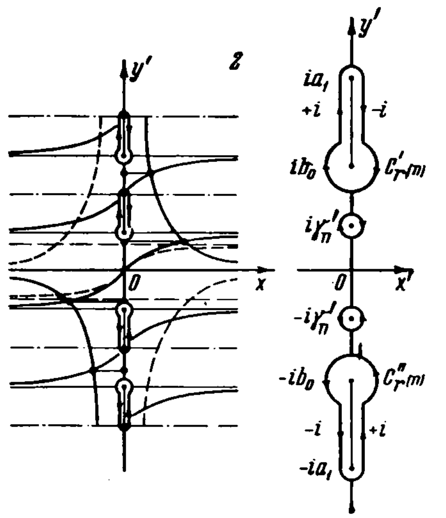

Fig. 4.

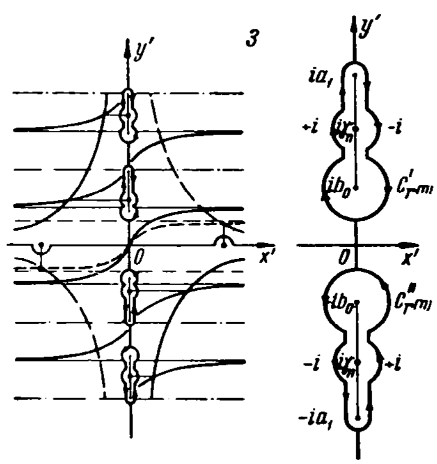

Fig. 5.

S_1, S_2, S_3 and S_4 the two cases are different from each other, since in this case the equation $kg \sinh(kh) + \xi^2 \cosh(kh) = 0$ has an infinite number of roots γ_n', where $n = 1, 2, 3, \ldots$, which lie on the imaginary axis, as shown in Fig. 4.

To calculate R_2^* we shall choose the same auxiliary functions and integration contours, only this time in Expressions (1.25) and (1.26) λ will be replaced by ξ. Contour (1) lies in plane 2 (Fig. 4), and contours (2) and (3) lie in plane 2′, which is related to plane 2 by the mapping of (x', y') onto $(-y', x')$ and a 90° clockwise rotation. Note these properties of the integrands, and after integration we have

$$\oint_{(1)2} F_1(z')\,dz' = 2R_2^* + iN_1 + iN_2 - L^* e^{-\xi t} + f_1^*(\alpha, Y, t) = 0 \tag{1.44}$$

$$\oint_{(2)2'} F_2(z')\,dz' = 2N_1 + 2N_2 + H_1 - iL^*(e^{\xi t} - e^{-\xi t}) + f_2^*(\alpha, Y, t) = 0$$

$$\oint_{(3)2'} F_3(z')\,dz' = -i2R_2^* + H_2 + iL^*(e^{\xi t} + e^{-\xi t}) + f_3^*(\alpha, Y, t) = 0$$

where N_1 and N_2 are the same symbols as in (1.30) except that λ is replaced by ξ

$$L^* = 2\pi \sum_{n=1}^{\infty} \frac{\gamma'_{nh^2} \cos[\gamma'_n (Y+h)/h]}{g(\alpha^2 h^2 - \gamma'^2_n)\,\Gamma(\gamma_n', \sigma) \cos^2 \gamma'_n}$$

$$\left(\Gamma(\gamma'_n, \sigma) = \gamma_n'^2 + (1+\sigma h) h\sigma, \quad \sigma = \frac{\xi^2}{g}\right)$$

which is obtained by means of integration over the small circles with centers at the points γ_n', whose radii tend to zero.

R_2^* is found from the system of equation (1.44). After a series of appropriate integrations we obtain

$$\frac{\partial\varphi}{\partial t} = 2\xi\sigma h^2 V_1 \sum_{n=1}^{\infty} \frac{\cos[\gamma'_n (Y+h)/h]}{\gamma_n' \Gamma(\gamma_n', \sigma) \cos\gamma_n'} e^{-\xi t} - 2\sigma h V_1 V_2 \sum_{n=1}^{\infty} \frac{\sin[\gamma'_n (Y+h)/h]}{\Gamma(\gamma'_n, \sigma)\cos\gamma_n'} e^{-2\xi t} -$$

$$- \xi V'_2 Y e^{-\xi t} - (V_2^2 + V_2^2) e^{-2\xi t} - U_{2} g(1 - e^{-\xi t}) \tag{1.45}$$

$$\frac{\partial\varphi}{\partial y} = 2\sigma h V_1 \sum_{n=1}^{\infty} \frac{\sin[\gamma_n' (Y+h)/h]}{\Gamma(\gamma_n', \sigma)\cos\gamma'_n} e^{-\xi t} + V_2 e^{-\xi t}, \qquad \frac{\partial\varphi}{\partial x} = V_1 e^{-\xi t}$$

for $t > 0$, $y < U_2(t)$, $x = U_1(t)$.

For the time $t = 0$ we have for $y < 0$, $x = 0$

$$\frac{\partial\varphi}{\partial t} = -\frac{2\xi}{h} V_1 \sum_{m=1,3}^{\infty} \frac{\sin CY}{C^2} - 2\sigma h V_1 V_2 \sum_{n=1}^{\infty} \frac{\sin[\gamma_n'(Y+h)/h]}{\Gamma(\gamma_n', \sigma)\cos\gamma_n'} - \xi V_2 Y - V_1^2 - V_2^2$$

$$\frac{\partial\varphi}{\partial y} = 2\sigma h V_1 \sum_{n=1}^{\infty} \frac{\sin[\gamma_n'(Y+h)/h]}{\Gamma(\gamma'_n, \sigma)\cos\gamma'_n} + V_2, \qquad \frac{\partial\varphi}{\partial x} = V_1 \tag{1.46}$$

2. Now we shall study the problem of the dynamic fluid pressure acting on a dam, which depends on the initial conditions

$$\varphi(x, y, 0) = 0, \qquad \frac{\partial\varphi(x, 0, 0)}{\partial t} = 0 \tag{2.1}$$

and the boundary conditions

$$\frac{\partial\varphi}{\partial x} = V_1 \sin\omega t \quad (V_1 = V_0\cos\vartheta) \quad \text{at } x = -U_1\cos\omega t \qquad (U_1 = U_0\cos\vartheta) \tag{2.2}$$

$$\frac{\partial\varphi}{\partial y} = V_2 \sin\omega t \quad (V_2 = V_0\sin\vartheta) \quad \text{at } y = -h - U_2\cos\omega t \qquad (U_2 = U_0\sin\vartheta) \tag{2.3}$$

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial y} = 0 \qquad \text{at } y = -U_2\cos\omega t \tag{2.4}$$

Here V_0 and U_0 are the amplitudes of the velocity and the displacement of the vibrating earth surface.

Let us choose the following form of the velocity potential $\phi(x, y, t)$ which satisfies $\Delta\phi = 0$:

$$\varphi(x, y, t) = \sin\omega t\left\{\int_0^\infty [B(k)\cosh k(Y+h) + C(k)\sinh kY]\cos kX\,dk + \int_0^\infty A(\alpha)\sin\alpha Y e^{-\alpha X} d\alpha\right\} + \int_0^\infty D(k)\cosh k(Y+h)\cos kX \sin Mt\,dk \quad \begin{pmatrix} X = x + U_1\cos\omega t \\ Y = y + U_2\cos\omega t \end{pmatrix} \tag{2.5}$$

Here the functions $A(a)$, $B(k)$, $C(k)$ and $D(k)$ are arbitrary.

With the aid of the Fourier integral, using conditions (2.1) to (2.4), we find

$$A(\alpha) = \frac{2V_1}{\pi}\frac{(1-\cos\alpha h)}{\alpha^2}, \qquad D(k) = -\frac{\omega}{M}B(k), \qquad M = \sqrt{kg\tanh kh}$$

$$C(k) = \frac{\delta(k)V_2}{k\cosh kh} - \frac{4V_1}{\pi^2}\int_0^\infty \frac{(1-\cos\alpha h)\cos\alpha h}{(\alpha^2+k^2)k\cosh kh}\,d\alpha \tag{2.6}$$

$$(kg \sinh kh - \omega^2 \cosh kh)\, B(k) + kg\, C(k) = -\frac{2g}{\pi}\int_0^\infty \frac{A(\alpha)\,\alpha^2}{\alpha^2 + k^2}\, d\alpha$$

Differentation of $\phi(x, y, t)$ with respect to t for $x = -U_1 \cos \omega t$ yields

$$\frac{\partial\varphi}{\partial t} = \frac{4V_1}{\pi^2}\iint_0^\infty \frac{(1-\cos\alpha h)\cos\alpha h}{(\alpha^2+k^2)\,k\cosh kh}(V_2 k \cosh kY \sin^2\omega t - \omega\cos\omega t \sinh kY)\, d\alpha\, dk +$$
$$+ \frac{4V_1}{\pi^2} g\,[\omega(S_5 - S_6) - V_2(S_7 - S_8)] + U_2 g(\cos\omega t - 1) + \omega V_2 Y \cos\omega t -$$
$$- V_2^2 \sin^2\omega t + \frac{2V_1}{\pi}\int_0^\infty \frac{(1-\cos\alpha h)}{\alpha^2}[\omega\cos\omega t \sin\alpha Y + \qquad (2.7)$$
$$+ \alpha \sin^2\omega t\,(V_1 \sin\alpha Y - V_2 \cos\alpha Y)]\, d\alpha$$

Here

$$S_5 = \cos\omega t \iint_0^\infty \Omega(\alpha, k)\cosh k(Y+h)\, d\alpha\, dk$$

$$S_6 = \iint_0^\infty \Omega(\alpha, k)\cosh k(Y+h)\cos Mt\, d\alpha\, dk$$

$$S_7 = \sin^2\omega t \iint_0^\infty \Omega(\alpha, k)\, k \sinh k(Y+h)\, d\alpha\, dk$$

$$S_8 = \omega\sin\omega t \iint_0^\infty \frac{\sin Mt}{M}\Omega(\alpha, k)\, k\sinh k(Y+h)\, d\alpha\, dk$$

$$\Omega(\alpha, k) = (1-\cos\alpha h)(\cos\alpha h - \cosh kh)/(\alpha^2+k^2)\times$$
$$\times (kg\sinh kh - \omega^2\cosh kh)\cosh kh$$

It should be noted that in the computation of the integrals S_6 and S_8 we first integrate with respect to k, and all remaining double integrals at first with respect to α. In the process of computing S_6 we analyze the integral

$$T^* = \int_0^\infty \frac{\cosh k(Y+h)\cos Mt\, dk}{(\alpha^2+k^2)(kg\sinh kh - \omega^2\cosh kh)\cosh kh} \qquad (2.8)$$

Let us take three auxiliary functions, similar to (1.26), when λ^2 is replaced by $-\omega^2$. Here the equation $kg \sinh(kh) - \omega^2 \cosh(kh) = 0$ has an infinite number of imaginary roots γ_n which fall onto the segments shown in Fig. 5, and two real roots.

We choose integration contours, as shown in Fig. 3, such that contour

(1) lies in plane 3 (Fig. 5), and contours (2) and (3) lie in plane 3′, which is obtained from plane 3 by the mapping of (x', y') onto $(-y', x')$ and a 90° clockwise rotation. By the theorem of residues we have

$$\begin{aligned} &\oint_{(1)3} F_1(z')\,dz' = 2T^* - iN_3 - iN_4 + K_1^* - K_2^* + f_1^*(\alpha, Y, t) = 0 \\ &\oint_{(2)3'} F_2(z')\,dz' = 2N_3 + 2N_4 + i2K_1^* + H_1 + f_2^*(\alpha, Y, t) = 0 \qquad (2.9) \\ &\oint_{(3)3'} F_3(z')\,dz' = i2T^* - i2K_2^* - iK_1^* + H_2 + f_3^*(\alpha, Y, t) = 0 \end{aligned}$$

where $N_3 = -N_1$, $N_4 = -N_2$, but instead of λ^2 we have $-\omega^2$,

$$K_1^* = 2\pi\gamma_s h^2 \frac{\cosh[\gamma_s(Y+h)/h]\sin\omega t}{g(\alpha^2h^2+\gamma_s^2)\Gamma(\gamma_s, Q)\cosh^2\gamma_s}$$

$$K_2^* = \frac{2\pi}{g}h^2\sum_{n=1}^{\infty}\frac{\gamma_n\cos[\gamma_n(Y+h)/h]\cos\omega t}{(\alpha^2h^2-\gamma_n^2)\Gamma(\gamma_n, Q)\cos^2\gamma_n}$$

$$\Gamma(\gamma_s, Q) = \gamma_s^2 + (1-Qh)Qh, \qquad \Gamma(\gamma_n, Q) = \gamma_n^2 - (1-Qh)Qh, \qquad Q = \omega^2/g$$

where K_2^* is obtained by means of integration over the small semicircles whose centers lie at the points γ_n, and whose radii tend to zero (Fig.5).

From the system (2.9) we find

$$T^* = -\frac{\pi}{g}h^2\frac{\gamma_s\cosh[\gamma_s(Y+h)/h]\sin\omega t}{(\alpha^2h^2+\gamma_s^2)\Gamma(\gamma_s, Q)\cosh^2\gamma_s} + f^*(\alpha, Y, t) \qquad (2.10)$$

After substitution of (2.10) into the expression for S_6 and integration with respect to α we obtain

$$S_6 = -\frac{\pi^2}{2g}Qh^2\frac{\cosh[\gamma_s(Y+h)/h]}{\gamma_s\Gamma(\gamma_s, Q)\cosh\gamma_s}\sin\omega t \qquad (2.11)$$

In a similar fashion we find

$$S_8 = \frac{\pi^2}{4g}Qh\frac{\sinh[\gamma_s(Y+h)/h]}{\Gamma(\gamma_s, Q)\cosh\gamma_s}\sin 2\omega t \qquad (2.12)$$

After evaluating the remaining double integrals and substituting them into (2.7) we have for $t > 0$, $Y < 0$, $x = -U_1\cos\omega t$

$$\frac{\partial\varphi}{\partial t} = 2\omega Qh^2V_1\sum_{n=1}^{\infty}\frac{\cos[\gamma_n(Y+h)/h]}{\gamma_n\Gamma(\gamma_n, Q)\cos\gamma_n}\cos\omega t + \qquad (2.13)$$

$$+ 2QhV_1V_2 \sum_{n=1}^{\infty} \frac{\sin[\gamma_n (Y+h)/h]}{\Gamma(\gamma_n, Q)\cos\gamma_n} \sin^2 \omega t -$$

$$- 2\omega Qh^2V_1 \frac{\cosh[\gamma_s (Y+h)/h]}{\gamma_s \Gamma(\gamma_s, Q)\cosh\gamma_s} \sin\omega t - QhV_1V_2 \frac{\sinh[\gamma_s (Y+h)/h]}{\Gamma(\gamma_s, Q)\cosh\gamma_s} \sin 2\omega t +$$

$$+ \omega V_2 Y \cos\omega t - U_2 g(1-\cos\omega t) - (V_1^2 + V_2^2)\sin^2\omega t$$

For $t > 0$, $Y < 0$, $x = -U_1 \cos\omega t$ we find (2.14)

$$\frac{\partial\varphi}{\partial y} = -2QhV_1 \sum_{n=1}^{\infty} \frac{\sin[\gamma_n (Y+h)/h]}{\Gamma(\gamma_n, Q)\cos\gamma_n} \sin\omega t +$$

$$+ 2QhV_1 \frac{\sinh[\gamma_s (Y+h)/h]}{\Gamma(\gamma_s, Q)\cosh\gamma_s} \cos\omega t + V_2 \sin\omega t$$

We know that $\partial\phi/\partial x = V_1 \sin\omega t$. For $t = 0$, $x = -U_1$ we have

$$\frac{\partial\varphi}{\partial t} = \frac{2\omega}{h} V_1 \sum_{m=1,3}^{\infty} \frac{\sin CY}{C^2} + \omega V_2 Y \qquad (2.15)$$

When we denote the dynamic fluid pressure by p^* and the fluid density by ρ we obtain the formula

$$\frac{p^*}{\rho} = -\frac{\partial\varphi}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]$$

Let us use p^* in the form $p_1^* + p_2^*$, i.e. $p^* = p_1^* + p_2^*$, where p_1^* denotes the set of components which does not contain the factor V_2, and p_2^* represents the remaining components. Formulas (1.37) and (1.38) show that p_1^* grows rapidly when ξ and η grow and $\xi \to \eta$, because then in $\cosh(\gamma) = \cosh(p)\cos q - i\sin(q)\sinh(p)$ the value of $p \to 0$ and $q \to 1/2\,\pi$.

It follows from Formula (1.45) that in this case p_1^* grows rapidly with a growing ξ, because then $\gamma_n' \to 1/2\,\pi$. Then the pressure reaches its maximum value at the time when the moving liquid meets with the instantly stationary dam. It can be seen from Formulas (2.13) and (2.14) that in this case p_1^* grows rapidly with an increase of ω, because then $\gamma_n \to 1/2\,\pi$.

It follows from the results obtained that vertical oscillations of the earth surface exhibit a significant influence on the loading of the dam during a destructive as well as during a strong earthquake. Actually, there may be some relations $U_2\omega^2 \geqslant g$, $U_2\xi^2 \geqslant g$, $U_2\eta^2 \geqslant g$ or $U_2\xi\eta \geqslant g$, and consequently p_2^* can be larger than the static pressure $p^\circ = \rho g Y$. The pressure p_1^* can also exceed the value of p° for some given η, ξ, ω. The formulas obtained above are useful for the construction of individual graphs of the distribution of the dynamic fluid pressure along a dam.

The problem of the dynamic fluid pressure on a dam which is caused by its vibrations according to $V = V_0 \cos \omega t$ was discussed in [4].

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